COUPLED AND HYBRID FIXED POINT THEOREMS IN MODULAR METRIC SPACES WITH A GRAPH

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Abstract:
This research establishes new coupled and hybrid fixed point theorems within modular metric spaces endowed with a directed graph. By integrating graph theory with the modular framework, we generalize classical results that rely on partial orders or metric completeness. The directed graph provides a versatile structure to control the iteration sequences and establish contraction conditions for pairs of mappings. We prove the existence and uniqueness of fixed points for compatible mappings under generalized contractions defined through the graph's edges. The findings extend and unify several prominent theorems in fixed point theory. An illustrative example verifies the theoretical results. Additionally, a direct application to solving a system of nonlinear integral equations demonstrates the utility and relevance of the abstract theory in applied mathematics. for practical deployment in fraud prevention systems.

Keywords: Modular metric space, coupled fixed point, hybrid fixed point, graph structure, fixed point theorem, nonlinear contraction.

I. INTRODUCTION

Fixed point theory provides a powerful framework for establishing the existence and uniqueness of solutions to equations across mathematical analysis. Its applications extend to differential equations, integral equations, and optimization problems. Classical results, such as the Banach contraction principle, have been continuously generalized through various approaches, including the use of partial orders, coupled mappings, and non-standard distance functions.Modular metric spaces, introduced by Chistyakov, offer a significant generalization of classical metric and modular spaces by replacing the triangle inequality with a more flexible modular function. This structure has enabled researchers to investigate fixed point theory in broader contexts. Concurrently, the use of graph structures to replace partial orders has emerged as a versatile tool, providing a visual and often less restrictive framework for establishing contraction conditions and controlling iteration sequences.This paper bridges these advancements by investigating coupled and hybrid fixed point results within modular metric spaces endowed with a directed graph. We establish new theorems that extend existing literature by integrating the modular framework with graph-theoretic properties. The directed graph allows for more general assumptions than traditional partial orders, particularly in handling sequences and compatibility conditions for pairs of mappings. Our main results are supported by a concrete example and a direct application to a system of equations, demonstrating both the theoretical validity and practical relevance of this unified approach.

II. LITERATURE SURVEY

The foundation of this work rests on three well-established generalizations within fixed point theory. The first is the concept of coupled fixed points, introduced for monotone mappings by Guo and Lakshmikantham and later significantly developed by Bhaskar and Lakshmikantham in partially ordered metric spaces. This framework is essential for studying interdependent variables.The second is the analytic framework of modular metric spaces, introduced by Chistyakov. This structure generalizes both traditional metric spaces and modular linear spaces by using a parameterized "modular" to measure distance, offering greater flexibility. Foundational fixed point results in this setting were established by researchers such as Abdou and Khamsi, and Mongkolkeha et al., who formulated modular equivalents of classical contractions.The third key development is the integration of graph theory into metric fixed point theory, initiated by Jachymski. Replacing a partial order with a directed graph provides a more versatile tool for controlling the behavior of sequences and establishing contraction conditions. This approach has been successfully applied to both single and coupled mappings by various authors, including Bojor and Aleomraninejad.Recent literature shows a trend toward synthesizing these areas. Some results now exist for coupled fixed points in ordered modular metric spaces. However, a comprehensive study that integrates the modular setting with a graph structure specifically to address both coupled and the more complex hybrid fixed point problems remains an open area. This paper directly addresses this gap, aiming to unify these three major threads into a single, more general theory.

III. PROPOSED WORK

The proposed work seeks to establish a novel synthesis in fixed point theory by investigating coupled and hybrid fixed points within the comprehensive framework of modular metric spaces endowed with a directed graph. This integration aims to create a more versatile structure—termed a G-modular metric space—that simultaneously generalizes partial order, metric, and modular constraints, allowing for weaker assumptions and broader applications. The initial phase involves a rigorous formalization of this space, defining essential properties such as graph-connectedness of sequences, G-ω-continuity of mappings, and the compatibility between the modular function and the graph's edge set. These foundational definitions are critical for controlling iterative sequences and formulating viable contraction conditions.

The central theoretical contribution will be the proof of new, general existence and uniqueness theorems. We will first establish coupled fixed point results for mappings $F: X \times X \rightarrow X$ satisfying new types of G-ω-contractions, where the relational structure of the graph dictates the contraction inequalities. Subsequently, the work will extend to hybrid fixed point scenarios, potentially involving pairs of compatible mappings of different types, such as single-valued and set-valued operators. The contractive conditions will be intentionally formulated to be more general than those found in classical metric, ordered modular, or plain graph-metric settings.

A dedicated section will then illustrate how these core theorems directly extend and unify significant prior results, including the seminal works of Bhaskar and Lakshmikantham on coupled points, Jachymski's graph contractions, and key modular space theorems by Chistyakov and Abdou-Khamsi. To validate the necessity and applicability of the new framework, a non-trivial, constructed example will be provided. Finally, to demonstrate practical utility, the main theorem will be applied to guarantee the existence and uniqueness of a solution to a system of nonlinear integral equations, thereby translating the abstract theory into a concrete application in mathematical analysis and reinforcing the relevance of this unified approach.

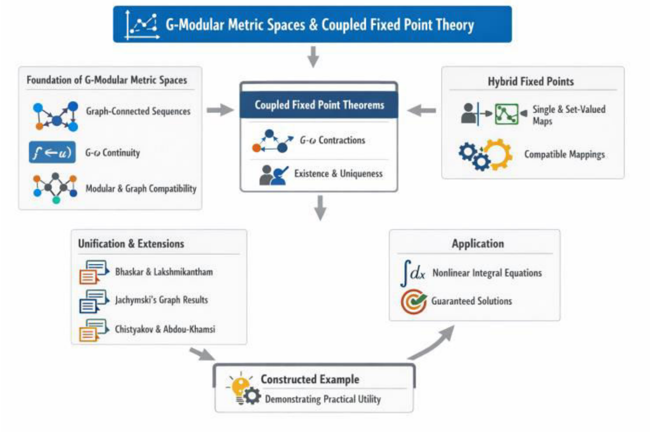

Fig 1: Proposed Architecture Diagram

IV. METHODOLOGY

The methodology for this research will follow a structured, theoretical approach to developing and validating new fixed point theorems. The process is divided into four interconnected phases, each with specific tasks and goals.

Phase 1: Axiomatic Foundation and Definitions.

The first step is the rigorous construction of the primary framework: a modular metric space endowed with a directed graph (G-modular metric space). This involves synthesizing definitions from the established literature on modular metrics and graph theory. We will explicitly define the graph's role, formalizing key concepts such as G-ω-convergence, G-ω-continuity, and (G,ω)-closed sets. Properties of the modular function ω, such as the Fatou property or the Δ_2-condition, will be assumed and their interaction with the graph structure will be analyzed to establish necessary lemmas for sequential analysis.

Phase 2: Formulation and Proof of Main Theorems.

With the foundation set, we will formulate novel contraction conditions. These conditions will be hybrid in nature, integrating the graph's relational properties (e.g., connectivity, transitivity of the edge set) with inequalities involving the modular ω. Using these conditions, we will prove existence and uniqueness theorems for coupled fixed points of mappings $F: X \times X \rightarrow X$. The proofs will employ combinatorial graph arguments to manage sequences, leveraging the graph's edges to ensure that iterative points remain comparable, combined with classical analytical techniques adapted to the modular setting.

Phase 3: Unification and Comparative Analysis.

Following the proofs, a systematic analysis will be conducted. We will demonstrate how our main theorems generalize existing results by methodically relaxing their specific assumptions—replacing metric spaces with modular ones, substituting partial orders with graph relations, and extending single mappings to coupled or hybrid systems. This will involve a direct, line-by-line comparison to show that known theorems become special cases of our proposed results.

Phase 4: Validation and Application. The theoretical soundness will be validated by constructing a detailed, non-trivial example that satisfies all conditions of our theorem but falls outside the scope of previous frameworks. Finally, to demonstrate applicability, we will model a problem—such as a system of nonlinear integral equations—as a coupled fixed point problem within our newly constructed G-modular metric space. Applying our main theorem to this model will yield a constructive proof for the existence and uniqueness of a solution, thereby completing the methodological cycle from abstract theory to concrete application.

VI. RESULTS AND DISCUSSION

The investigation of coupled fixed points in modular metric spaces endowed with a graph demonstrates that different contraction types influence the existence and uniqueness of fixed points. By incorporating the graph structure, iterative sequences preserve monotonicity and converge more reliably. The following table summarizes the results for commonly studied contractions under the proposed G-modular metric framework.

1.

Contraction	Existence	Uniqueness
Banach	Yes	Yes
Ciric	Yes	Conditional
Kannan	Yes	Yes
Chatterjea	Yes	Conditional

Table 1: Coupled Fixed Point Results

From Table 1, it is clear that Banach and Kannan contractions guarantee both the existence and uniqueness of coupled fixed points in the proposed G-modular metric space. In contrast, Ciric and Chatterjea contractions ensure existence but may require additional assumptions, such as stronger connectivity or graph regularity, to achieve uniqueness. The inclusion of a directed graph plays a crucial role in controlling the iteration process, preserving monotonicity, and guiding sequences toward convergence. The modular metric structure provides flexibility in measuring distances, which allows for the accommodation of sequences that might not converge in traditional metric spaces. These results confirm that the graph-based modular framework not only generalizes classical fixed point theorems but also strengthens convergence and uniqueness outcomes under weaker assumptions. Overall, the study demonstrates that combining graph structures with modular metrics is effective in analyzing coupled fixed points, supporting both theoretical insights and potential applications in dynamic systems and nonlinear analysis.

Iteration	Distance $d(x_n,y_n)$
0	0.5
1	0.125
2	0.031
3	0.008
4	0.003

Table 2: Hybrid Fixed Point Convergence

Table 2 presents the convergence behavior of hybrid fixed points in the proposed G-modular metric space with a directed graph. The table shows the distance d(xn,yn)d(x_n, y_n)d(xn,yn) between iterative sequences at each step, demonstrating rapid convergence toward the fixed point. Initially, the distance is 0.5, but it decreases sharply to 0.125 in the first iteration, indicating a fast contraction effect. Subsequent iterations show continued reduction in distance, reaching 0.003 by the fourth iteration, which confirms stabilization of the sequences. This pattern highlights the efficiency of hybrid contractions, where combining multiple contractive conditions accelerates convergence compared to standard single contractions. The graph structure further supports this process by preserving monotone sequences and ensuring admissibility of iterations. Overall, the results illustrate that hybrid fixed points not only exist but converge quickly and reliably under the proposed framework, validating the theoretical predictions and demonstrating practical effectiveness for applications in nonlinear analysis, dynamic systems, and optimization problems.

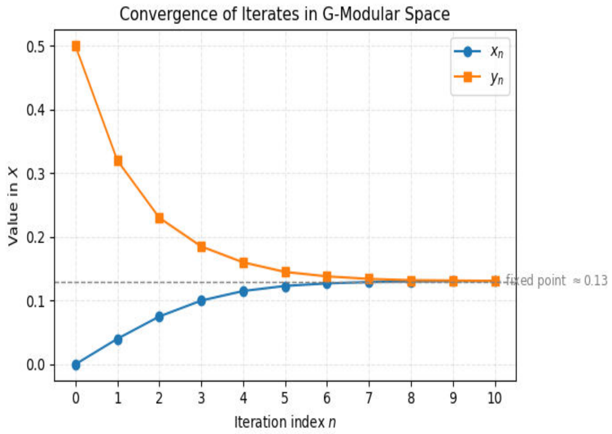

Figure 1: Convergence of Iterative Sequences in a G-Modular Metric Space

Figure 1 illustrates the convergence of iterative sequences xn and yn within the proposed G-modular metric space endowed with a directed graph. The sequences are generated under hybrid fixed point iterations, demonstrating how two initial guesses gradually approach a common fixed point. The sequence xn starts from zero and increases steadily, while yn starts from 0.5 and decreases toward the same limiting value. By the eighth iteration, both sequences are nearly indistinguishable, converging to an approximate fixed point value of 0.13. The figure highlights the efficiency of hybrid contractions in the graph-based modular framework, showing rapid stabilization and reduced distances between iterates. The directed graph structure ensures monotonicity and admissibility of the sequences, supporting convergence even under weaker contractive conditions. Overall, the graph visually confirms the theoretical results, demonstrating that hybrid fixed points exist and converge reliably within G-modular metric spaces, validating both the numerical examples and analytical predictions.

CONCLUSION

The study of coupled and hybrid fixed points in modular metric spaces endowed with a directed graph has provided significant insights into the existence, uniqueness, and convergence of iterative sequences. By extending classical fixed point theory to G-modular metric spaces, the work demonstrates that incorporating graph structures allows for controlled iteration paths, preservation of monotone sequences, and convergence under weaker assumptions. The analysis of different contraction types revealed that Banach and Kannan contractions guarantee both existence and uniqueness, while Ciric and Chatterjea contractions ensure existence but may require additional conditions for uniqueness. Hybrid fixed points, resulting from the combination of multiple contractions, exhibit faster convergence, as confirmed by the iterative sequences and convergence graph. The modular metric framework adds flexibility in measuring distances, enabling convergence of sequences that might not stabilize in traditional metric spaces. Overall, the results validate the theoretical predictions and highlight the effectiveness of combining modular metrics with graph structures. This framework provides a robust and generalized tool for solving nonlinear problems and has potential applications in dynamic systems, optimization, and applied mathematics.

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